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Let x,y,z be real numbers such that $x^2+y^2+z^2=9$. Prove that $2(x+y+z)-xyz \le 10$.

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First we will prove inequality that for any real $x,y,z \ge 0$ such that $x^2+y^2+z^2=9$ holds inequality $2(x+y+z)-xyz \le 6\sqrt{2}$ $(6\sqrt{2}<10)$.

Let $s := x+y+z, p := xy+yz+zx, q := xyz$.

Since $9q \ge 4sp - s^3$ (Schur's Inequality $\sum x(x-y)(x-z) \ge 0$) and

$9 = x^2+y^2+z^2 = s^2-2p \iff 2p = s^2-9$ then $9q \ge 2s(s^2-9)-s^3 = s^3-18s$

and, therefore, $q \ge q_* = \max\left\{0, \dfrac{s^3-18s}{9}\right\}$.

Noting that $27 = 3(x^2+y^2+z^2) \ge (x+y+z)^2 = s^2 \iff s \le 3\sqrt{3}$, we obtain:

1. $q_* = \dfrac{s^3-18s}{9}$ for $s \in \left[3\sqrt{2}, 3\sqrt{3}\right]$ and, therefore, for such s we have

$$6\sqrt{2}-(2(x+y+z)-xyz) = 6\sqrt{2}-2s+q \ge 6\sqrt{2}-2s+q_* = 6\sqrt{2}-2s+\frac{s^3-18s}{9} =$$
$$\frac{\left(s-3\sqrt{2}\right)\left(3\sqrt{2}s+s^2-18\right)}{9} \ge \frac{\left(s-3\sqrt{2}\right)(s^2-18)}{9} = \frac{\left(s-3\sqrt{2}\right)^2\left(s+3\sqrt{2}\right)}{9} \ge 0;$$

2. $q_* = \dfrac{s^3-18s}{9}$ for $s \in (0, 3\sqrt{2}]$ and, therefore,for such s we have

$$6\sqrt{2}-(2(x+y+z)-xyz) = 6\sqrt{2}-2s+q \ge 6\sqrt{2}-2s = 2\left(3\sqrt{2}-s\right) > 0.$$

Consider now two cases:

1. If $xyz > 0$ then

$$2(x+y+z)-xyz \le 2(|x|+|y|+|z|)-|xyz| = 2(|x|+|y|+|z|)-|x|\cdot|y|\cdot|z| < 10$$

by considered above case $x,y,z \ge 0$ (because $|x|^2+|y|^2+|z|^2 = x^2+y^2+z^2 = 9$);

2. If $xyz < 0$ then at least one of the numbers x,y,z is negative, let it be z.

Then $xy > 0, z = -|z|$ and, therefore, $2(x+y+z)-xyz \le 2(|x|+|y|)-2|z|+|x|\cdot|y|\cdot|z|$.

Let $f(x,y,z) := 2(x+y)-2z+xyz$, where $x,y,z > 0$ and $x^2+y^2+z^2=9$.

We will prove that for such x,y,z holds $\max f(x,y,z) = 10$.

Since $x+y \le \sqrt{2(x^2+y^2)}, xy \le \dfrac{x^2+y^2}{2}$ then $f(x,y,z) \le 2\sqrt{2(x^2+y^2)}-2z+z\cdot\dfrac{x^2+y^2}{2} =$

$2\sqrt{2(9-z^2)}-2z+z\cdot\dfrac{9-z^2}{2} = 2\sqrt{2(9-z^2)}-\dfrac{z^3}{2}+\dfrac{5z}{2}$.

Let $h(z) := 2\sqrt{2(9-z^2)}-\dfrac{z^3}{2}+\dfrac{5z}{2}$, where $z \in (0,3)$.

Since $h'(z) = -2\sqrt{2}\dfrac{z}{\sqrt{9-z^2}}-\dfrac{3z^2}{2}+\dfrac{5}{2}$ strictly decrease on $(0,3)$ and $h'(1) = 0$

then $h'(z) > 0$ on $(0,1), h'(z) < 0$ on $(1,3)$ and, therefore,

$$\max\{h(z) \mid z \in (0,3)\} = h(1) = 2\sqrt{2(9-1^2)}-\frac{1^3}{2}+\frac{5\cdot 1}{2} = 10.$$

Hence, $2(x+y+z)-xyz \le f(|x|,|y|,|z|) \le 10$ and equality occurs iff $x=y=2, z=-1$.

Combining both considered cases we can conclude that

$2(x+y+z)-xyz \le 10$ for any real $x,y,z \in \mathbb{R}$ such that $x^2+y^2+z^2=9$

with equality iff $(x,y,z) \in \{(2,2,-1),(2,-1,2),(-1,2,2)\}$.